4. Absolute continuity and singularity. *Reading material:* Section 5.3.3 of [**Dur**] and section 1.4 of [**MP**].

Many processes can be built out of Brownian motion, for example, *Brownian bridge*, *Brownian motion conditioned to stay above* -1 *up to time* 1, etc. These tweak the behaviour of Brownian motion in a global way, but it seems reasonable that locally they behave the same as BM. In particular, nowhere differentiability or Hölder continuity, properties of zero sets etc. should hold for these processes too. It is possible but painful to adopt the proof in each new case. Instead, one could use the idea that these processes are mutually absolutely continuous to Brownian motion, which implies that all almost sure properties are preserved! It is best to read about absolute continuity and singularity of sequences of independent random variables first and the Cameron-Martin theorem is seen to fall in the general framework of things.

19. Let θ_p denote $\mathrm{Ber}(p)$ measure. Let $\nu = \otimes_{n=1}^{\infty}\theta_{1/2}$ and $\mu = \otimes_{n=1}^{\infty}\theta_{p_n}$, where $p_n = \frac{1}{2} + n^{-\gamma}$ for some fixed $\gamma > 0$. Apply Kakutani's criterion to determine whether μ is absolutely continuous or singular to ν. In case of singularity, can you give a simple "test criterion"?

20. Let $\nu = \otimes_{n=1}^{\infty}N(0,1)$ and $\mu = \otimes_{n=1}^{\infty}N(0,1+n^{-\gamma})$. Apply Kakutani's criterion to determine whether μ is absolutely continuous or singular to ν.

21. Let μ be a probability measure on $(\Omega, \mathcal{F}, \mathbf{P})$, $A \in \mathcal{F}$ and $\nu = \mu(\cdot\,|A)$. If $\mu(A) = 0$, then $\nu \perp \mu$ while if $\mu(A) > 0$, then ν is absolutely continuous to μ (but μ is not necessarily absolutely continuous to ν).

22. Let W be standard Brownian motion on I and let W_0 be standard Brownian bridge. Then $W \perp W_0$. However, $(W_0(t))_{t\leq 1-\delta}$ and $(W_t)_{t\leq 1-\delta}$ are mutually absolutely continuous for any $\delta > 0$.

[**Hint:** Let μ_n (respectively ν_n) be the distribution of $(W_{t_{n,1}}, \dots, W_{t_{n,2^n}})$ (respectively, $(W_0(t_{n,1}), \dots, W_0(t_{n,2^n}))$. Find the Radon-Nikodym of μ_n with respect to ν_n].

[**Dur**] will indicate Durrett's excellent book *Probability: theory and examples*, 4th edition, available at http://www.math.duke.edu/˜rtd/PTE/pte.html